Exercise 10

Use power series to solve the differential equation.

$$y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x^2 inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Make the substitution $n = k + 4$ in the first series and the substitution $n = k$ in the second series.

$$\sum_{k+4=2}^{\infty} (k+4)[(k+4)-1] a_{k+4} x^{(k+4)-2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0$$

Simplify the first sum.

$$\sum_{k=-2}^{\infty} (k+4)(k+3) a_{k+4} x^{k+2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0$$

Write out the first few terms of the first sum.

$$(2)(1)a_2 + (3)(2)a_3 x + \sum_{k=0}^{\infty} (k+4)(k+3) a_{k+4} x^{k+2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0$$

Combine the series.

$$(2)(1)a_2 + (3)(2)a_3 x + \sum_{k=0}^{\infty} \left[(k+4)(k+3) a_{k+4} + a_k\right] x^{k+2} = 0$$

a_2, a_3, and the quantity in square brackets must be zero.

$$(k+4)(k+3)a_{k+4} + a_k = 0 \qquad a_2 = 0 \qquad a_3 = 0$$

Solve for a_{k+4}.

$$a_{k+4} = -\frac{1}{(k+4)(k+3)}a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k=0: \quad a_4 = -\frac{1}{(0+4)(0+3)}a_0 = -\frac{1}{4\cdot 3}a_0$$

$$k=1: \quad a_5 = -\frac{1}{(1+4)(1+3)}a_1 = -\frac{1}{5\cdot 4}a_1$$

$$k=2: \quad a_6 = -\frac{1}{(2+4)(2+3)}a_2 = 0$$

$$k=3: \quad a_7 = -\frac{1}{(3+4)(3+3)}a_3 = 0$$

$$k=4: \quad a_8 = -\frac{1}{(4+4)(4+3)}a_4 = -\frac{1}{8\cdot 7}\left(-\frac{1}{4\cdot 3}a_0\right) = (-1)^2\frac{1}{8\cdot 7\cdot 4\cdot 3}a_0$$

$$k=5: \quad a_9 = -\frac{1}{(5+4)(5+3)}a_5 = -\frac{1}{9\cdot 8}\left(-\frac{1}{5\cdot 4}a_1\right) = (-1)^2\frac{1}{9\cdot 8\cdot 5\cdot 4}a_1$$

$$k=6: \quad a_{10} = -\frac{1}{(6+4)(6+3)}a_6 = 0$$

$$k=7: \quad a_{11} = -\frac{1}{(7+4)(7+3)}a_7 = 0$$

$$k=8: \quad a_{12} = -\frac{1}{(8+4)(8+3)}a_8 = -\frac{1}{12\cdot 11}\left[-(-1)^2\frac{1}{8\cdot 7\cdot 4\cdot 3}a_0\right] = (-1)^3\frac{1}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}a_0$$

$$k=9: \quad a_{13} = -\frac{1}{(9+4)(9+3)}a_9 = -\frac{1}{13\cdot 12}\left[(-1)^2\frac{1}{9\cdot 8\cdot 5\cdot 4}a_1\right] = (-1)^3\frac{1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}a_1$$

$$\vdots$$

The general formula is

$$a_{4m} = (-1)^m\frac{1}{(4m)(4m-1)(4m-4)(4m-5)\cdots 8\cdot 7\cdot 4\cdot 3}a_0$$

$$a_{4m+1} = (-1)^m\frac{1}{(4m+1)(4m)(4m-3)(4m-4)\cdots 9\cdot 8\cdot 5\cdot 4}a_1$$

$$a_{4m+2} = 0$$

$$a_{4m+3} = 0.$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \sum_{m=1}^{\infty} a_{4m} x^{4m} + \sum_{m=1}^{\infty} a_{4m+1} x^{4m+1} + \sum_{m=1}^{\infty} a_{4m+2} x^{4m+2} + \sum_{m=1}^{\infty} a_{4m+3} x^{4m+3} \\
&= a_0 + a_1 x + \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m)(4m-1)(4m-4)(4m-5)\cdots 8\cdot 7\cdot 4\cdot 3} a_0 x^{4m} \\
&\qquad + \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m+1)(4m)(4m-3)(4m-4)\cdots 9\cdot 8\cdot 5\cdot 4} a_1 x^{4m+1} \\
&\qquad\qquad + \sum_{m=1}^{\infty} (0) x^{4m+2} + \sum_{m=1}^{\infty} (0) x^{4m+3} \\
&= a_0 + a_1 x + a_0 \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m)(4m-1)(4m-4)(4m-5)\cdots 8\cdot 7\cdot 4\cdot 3} x^{4m} \\
&\qquad + a_1 \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m+1)(4m)(4m-3)(4m-4)\cdots 9\cdot 8\cdot 5\cdot 4} x^{4m+1},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. Differentiate it with respect to x.

$$\begin{aligned}
y'(x) = a_1 &+ a_0 \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m)(4m-1)(4m-4)(4m-5)\cdots 8\cdot 7\cdot 4\cdot 3} (4m x^{4m-1}) \\
&+ a_1 \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m+1)(4m)(4m-3)(4m-4)\cdots 9\cdot 8\cdot 5\cdot 4} (4m+1) x^{4m}
\end{aligned}$$

Apply the initial conditions to determine a_0 and a_1.

$$y(0) = a_0 = 1$$

$$y'(0) = a_1 = 0$$

Therefore,

$$y(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1}{(4m)(4m-1)(4m-4)(4m-5)\cdots 8\cdot 7\cdot 4\cdot 3} x^{4m}.$$